## 1 4th Year Materials Engineering

# Mechanics of Composite Materials – Lecture 3

## 2 Last Week

### 2.1 Summary

Transformations and symmetry, mirror and rotation

$$\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \beta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Stress and Strain

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \qquad \epsilon_{ij} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{bmatrix}$$

Briefly introduced the stiffness tensor

$$c_{ijkl}, \quad i, j, k, l \text{ take values in 1,2,3} \Rightarrow \text{81 entries}$$

## 3 Stiffness Tensor

$$c_{ijkl} \quad \text{Stiffness Tensor } 3 \times 3 \times 3 \times 3$$

Explanation

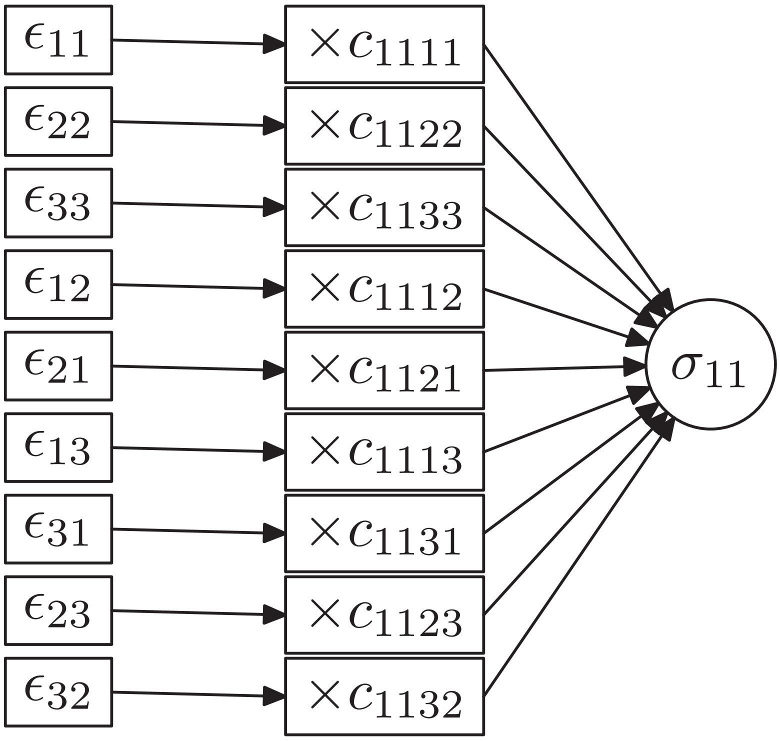
- 9 Strain components $\epsilon_{kl}$
- 9 Stess components $\sigma_{ij}$
- Every strain contributes to every stress (**In General**)
- $\Rightarrow 9 \times 9 = 81$ components of stiffness

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl}$$

Tensor Summation Convention... sum over repeated indices

$$\begin{aligned}\sigma_{ij} = {} & c_{ij11}\epsilon_{11} + c_{ij12}\epsilon_{12} + c_{ij13}\epsilon_{13} + c_{ij21}\epsilon_{21} + c_{ij22}\epsilon_{22} \\ & + c_{ij23}\epsilon_{23} + c_{ij31}\epsilon_{31} + c_{ij32}\epsilon_{32} + c_{ij33}\epsilon_{33}\end{aligned}$$

## 4 Diagram




And so on for the 8 other stresses.

## 5 Simplifications

Recall stress and strain tensors are symmetric.

$$\sigma_{ij} = \sigma_{ji} \qquad \epsilon_{ij} = \epsilon_{ji}$$

As a consequence

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{jilk}$$

Energy considerations lead to the following additional condition

$$c_{ijkl} = c_{klij}$$

$\Rightarrow$ **at most 21 independent stiffness constants**.

As an aside, note that we can also use **compliance**, which is the inverse of stiffness

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl}$$
$$\epsilon_{ij} = s_{ijkl}\sigma_{kl}$$

The first symmetry condition (to do with stress and strain) means taht there are only 6 stresses and 6 strains (not 9 of each), so we need only $6 \times 6 = 36$ stiffness constants. The second condition, to do with energy, eliminates a further 15 constants. This is equivalent to enforcing a symmetry condition on the following $6 \times 6$ matrix

$$\begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & c_{1123} & c_{1113} & c_{1112} \\ c_{2211} & c_{2222} & c_{2233} & c_{2223} & c_{2213} & c_{2212} \\ c_{3311} & c_{3322} & c_{3333} & c_{3323} & c_{3313} & c_{3312} \\ c_{2311} & c_{2322} & c_{2333} & c_{2323} & c_{2313} & c_{2312} \\ c_{1311} & c_{1322} & c_{1333} & c_{1323} & c_{1313} & c_{1312} \\ c_{1211} & c_{1222} & c_{1233} & c_{1223} & c_{1213} & c_{1212} \end{bmatrix}$$

Which then becomes:

$$\begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & c_{1123} & c_{1113} & c_{1112} \\ c_{1122} & c_{2222} & c_{2233} & c_{2223} & c_{2213} & c_{2212} \\ c_{1133} & c_{2233} & c_{3333} & c_{3323} & c_{3313} & c_{3312} \\ c_{1123} & c_{2223} & c_{3323} & c_{2323} & c_{2313} & c_{2312} \\ c_{1113} & c_{2213} & c_{3313} & c_{2313} & c_{1313} & c_{1312} \\ c_{1112} & c_{2212} & c_{3312} & c_{2312} & c_{1312} & c_{1212} \end{bmatrix}$$

and has only 21 independent entries (the diagonal 6 and the 15 entries above it).

# 6 Simplifications

How does this affect $\boxed{\sigma_{ij} = c_{ijkl}\epsilon_{kl}}$? From before:

$$\sigma_{ij} = c_{ij11}\epsilon_{11} + c_{ij12}\epsilon_{12} + c_{ij13}\epsilon_{13} + c_{ij21}\epsilon_{21} + c_{ij22}\epsilon_{22} + c_{ij23}\epsilon_{23} + c_{ij31}\epsilon_{31} + c_{ij32}\epsilon_{32} + c_{ij33}\epsilon_{33}$$

Apply symmetry to $c_{ijkl}$ and to $\epsilon_{kl}$ terms:

$$\sigma_{ij} = c_{ij11}\epsilon_{11} + c_{ij22}\epsilon_{22} + c_{ij33}\epsilon_{33} + 2c_{ij23}\epsilon_{23} + 2c_{ij13}\epsilon_{13} + 2c_{ij12}\epsilon_{12}$$

Now we introduce a new term: $\gamma$ is called the **engineering shear strain** and is simply defined as

$$\gamma_{ij} = 2\epsilon_{ij}, \text{ where } i \neq j. \tag{1}$$

Using this…

$$\boxed{\sigma_{ij} = c_{ij11}\epsilon_{11} + c_{ij22}\epsilon_{22} + c_{ij33}\epsilon_{33} + c_{ij23}\gamma_{23} + c_{ij13}\gamma_{13} + c_{ij12}\gamma_{12}}$$

# 7 Simplifications – Matrix Relation

Introduce **Reduced Matrix Notation**:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix}$$

Notation has changed a little for the stiffness, now our indexes go up to 6, though there are only two of them. The conversion is shown in the table.

Here, stress and strain look like vectors of length 6, and stiffness is just a square matrix. This has the advantage of being compact and easy to write. The disadvantage is that coordinate transformations are harder to do, and it is not as easy to program a computer to deal with this form. Note that the subscripts on the stress and strain have not been changed. They are just the same as before. Only stiffness has been altered in this way.

The reson for this notation is that it allows us to fully describe even a very general material using a notation that can be easily written in a single equation. Also, the contracted subscripts facilitate writing.

Note that the material properties shown, comprising 21 independent numbers, are sufficient to describe the linear elastic behaviour of even the most generally anisotropic material. Of course, there is much more that one could say

| Tensor subscript | | Matrix subscript |
|---|---|---|
| 11 | $\Rightarrow$ | 1 |
| 22 | $\Rightarrow$ | 2 |
| 33 | $\Rightarrow$ | 3 |
| 23 | $\Rightarrow$ | 4 |
| 13 | $\Rightarrow$ | 5 |
| 12 | $\Rightarrow$ | 6 |

about a material's other properties that we have not mentioned at all: density, viscoelastic behaviour, temperature response, etc.,

# 8 Simplifications – Material Symmetry

Up to now we have not required the material to have any symmetry. Although the stress, strain, and stiffness tensors have symmetry, this is not due to material symmetry, it is a consequence of equilibrium and energy considerations and has nothing to do with the material properties in question.

## 8.1 Stiffness Transformation

The stiffness tensor transforms like other tensors.

$$\begin{aligned} c'_{ijkl} &= \beta_{im}\beta_{jn}\beta_{ko}\beta_{lp}c_{mnop} \\ &= \sum_{m=1}^{3}\sum_{n=1}^{3}\sum_{o=1}^{3}\sum_{p=1}^{3}\beta_{im}\beta_{jn}\beta_{ko}\beta_{lp}c_{mnop} \end{aligned}$$

Remember, tensor summation convention. Sum over repeated indices. This is written explicitly on the second line of the equation. In the most general situation, all 81 entries in $c$ will contribute to each entry in $c'$. However, if you look at the transformation matrices we mentioned already, you'll notice several entries are usually zero. This simplifies the calculation somewhat as if even one of the $\beta$ terms is zero, then the entire product goes to zero.

Expanding for one entry in $c'$

$$\begin{aligned} c'_{1213} =& \beta_{11}\beta_{21}\beta_{11}\beta_{31}c_{1111} \\ &+ \beta_{11}\beta_{21}\beta_{11}\beta_{32}c_{1112} \\ &+ \beta_{11}\beta_{21}\beta_{11}\beta_{33}c_{1113} \\ &+ \beta_{11}\beta_{21}\beta_{12}\beta_{31}c_{1121} \\ &+ \beta_{11}\beta_{21}\beta_{12}\beta_{32}c_{1122} \\ &+ \beta_{11}\beta_{21}\beta_{12}\beta_{33}c_{1123} \\ &+ \ldots \text{75 more terms} \end{aligned}$$

The terms of $\beta$ depend on what transformation we are using. It might be a mirror reflection, or a rotation, or something else. If the material properties are symmetric under the transformation, then they are the same when transformed as they were before

# 9 Simplifications – Material Symmetry

Symmetry under $\beta$ means:

$$c'_{ijkl} = c_{ijkl}$$

$\Rightarrow$ set of equations. When we solve the equations, we will see that some of the stiffness constants will have to be zero to satisfy this requirement. Additionally, we may see that some stiffness constants need to be the same as others or are the same as some combination of other properties.

We have seen the transformation tensor for this already, many times. For clarity, it is repeated here:

$$\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Take each unique term from the $(6 \times 6)$ stiffness matrix. Write it down using full (4) tensor subscripts. Then write down the sum of terms that contribute to this term under the transformation. Pay attention to the position of the zeros in $\beta$ as this makes things much easier. Formulate a set of equations, then solve. The procedure is much the same as the one we applied to a single vector in the previous lecture

### 9.1 Example terms, $x_1$–$x_2$ Plane Mirror Symmetry

$$c_{ijkl} = c'_{ijkl} = \beta_{im}\beta_{jn}\beta_{ko}\beta_{lp}c_{mnop}$$

$$\begin{aligned} C_{11} \equiv c_{1111} = c'_{1111} &= \beta_{11}\beta_{11}\beta_{11}\beta_{11}c_{1111} + 0 \ldots \\ &= (1)(1)(1)(1)c_{1111} = c_{1111} \qquad \ldots \text{no effect} \end{aligned}$$

Note that we end up with just a single term in our summation. This is because the transformation matrix is **diagonal**. The other 80 terms in the summation all go to zero (which is very convenient).

## 10 Simplifications – Material Symmetry

$$c_{ijkl} = c'_{ijkl} = \beta_{im}\beta_{jn}\beta_{ko}\beta_{lp}c_{mnop}$$

$$\begin{aligned} C_{44} \equiv c_{2323} = c'_{2323} &= \beta_{22}\beta_{33}\beta_{22}\beta_{33}c_{2233} + 0 \ldots \\ &= (+1)(-1)(+1)(-1)c_{2233} = c_{2233} \quad \ldots \text{again, no effect} \end{aligned}$$

But some constants **are** affected...

$$\begin{aligned} C_{34} \equiv c_{3323} = c'_{3323} &= \beta_{33}\beta_{33}\beta_{22}\beta_{33}c_{3323} + 0 \ldots \\ &= (-1)(-1)(+1)(-1)c_{1111} = -c_{3323} \\ \Rightarrow C_{34} \equiv c_{3323} = 0 \end{aligned}$$

Note odd number of “3”s in subscript. We apply this repeatedely for the rest of the material’s stiffness constants, and we find that quite a few terms will go to zero (all the terms with an odd number of ”3”s in their subscripts... $c_{1113}$, $c_{1123}$, $c_{1312}$, $c_{1333}$, etc., To make an orthotropic material, we apply a second plane of symmetry and combine the effect of the two. Theoretically we need to apply a third perpendicular plane of symmetry, but because the stiffness tensor has an even order (4), this third plane actually imposes no extra restrictions on our stiffness tensor.

# 11 Single Symmetry Plane – Stiffness Matrix

## 11.1 $x_1$–$x_2$ Plane of Symmetry

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16}\\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26}\\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36}\\ 0 & 0 & 0 & C_{44} & C_{45} & 0\\ 0 & 0 & 0 & C_{45} & C_{55} & 0\\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66}\end{bmatrix} \begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ \gamma_{23}\\ \gamma_{13}\\ \gamma_{12}\end{bmatrix}$$

# 12 Orthotropic – Stiffness Matrix

## 12.1 3 Perpendicular Planes of Symmetry

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0\\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66}\end{bmatrix} \begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ \gamma_{23}\\ \gamma_{13}\\ \gamma_{12}\end{bmatrix}$$

Note:

- The planes are aligned along the coordinate axes.
- 9 independent elastic constants

# 13 Transversely Isotropic – Stiffness Matrix

## 13.1 Axis of Rotational Symmetry ($x_3$ axis)

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0\\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{55} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11}-C_{12})\end{bmatrix} \begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ \gamma_{23}\\ \gamma_{13}\\ \gamma_{12}\end{bmatrix}$$

Note:

- The axis is aligned along a coordinate direction
- 5 independent elastic constants

Note, that if you rotated the material so that the axis of symmetry was not aligned along the coordinate direction, then the stiffness matrix would not look as neat. There would not be as many zeros there. However, there would still be only 5 (or 9 for orthotropic) independent numbers. The other terms would be combinations of these numbers.

Also, you should note how the shear and normal components of stress/strain are decoupled. Normal stresses produce only normal strains and vice versa. Equally, shear strain gives rise only to shear stress.

# 14 Isotropic – Stiffness Matrix

## 14.1 Two Perpendicular Axes of Rotational Symmetry

$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{23}\\\sigma_{13}\\\sigma_{12}\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{12} & 0 & 0 & 0\\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0\\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2}(C_{11}-C_{12}) & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11}-C_{12}) & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11}-C_{12})\end{bmatrix} \begin{bmatrix}\epsilon_{11}\\\epsilon_{22}\\\epsilon_{33}\\\gamma_{23}\\\gamma_{13}\\\gamma_{12}\end{bmatrix}$$

Note:

- The axes can be aligned any way at all
- Just 2 independent elastic constants

# 15 Recap

We have discussed the following:

- Symmetry and coordinate transformations
- Detailed constitutive relations
  - Stress
  - Strain
  - Stiffness
- Form of Stiffness tensor, and how to write it
- Transformation of stiffness tensor & Material Symmetry
  - Orthotropic
  - Transversely Isotropic
  - Isotropic

# 16 Plates – Simplifications

## 16.1 Composite plates

Composites are often used in **thin plate-like components**

$\Rightarrow$ Plane Stress Conditions

i.e. All stress is in plane. No out of plane stress.

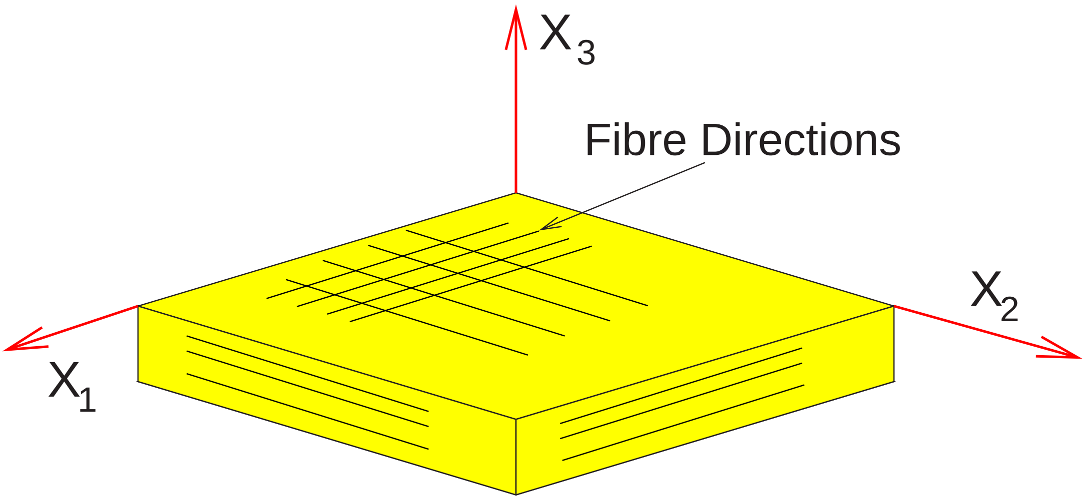
- Align plane with $x_1$–$x_2$ plane
- Nonzero stresses: $\sigma_{11}$, $\sigma_{22}$, $\sigma_{12}$
- Zero stresses: $\sigma_{33}$, $\sigma_{13}$, $\sigma_{23}$

# 17 Plates – Simplifications

## 17.1 Orthotropic

If laminate is Orthotropic (cross-ply laminate), we get

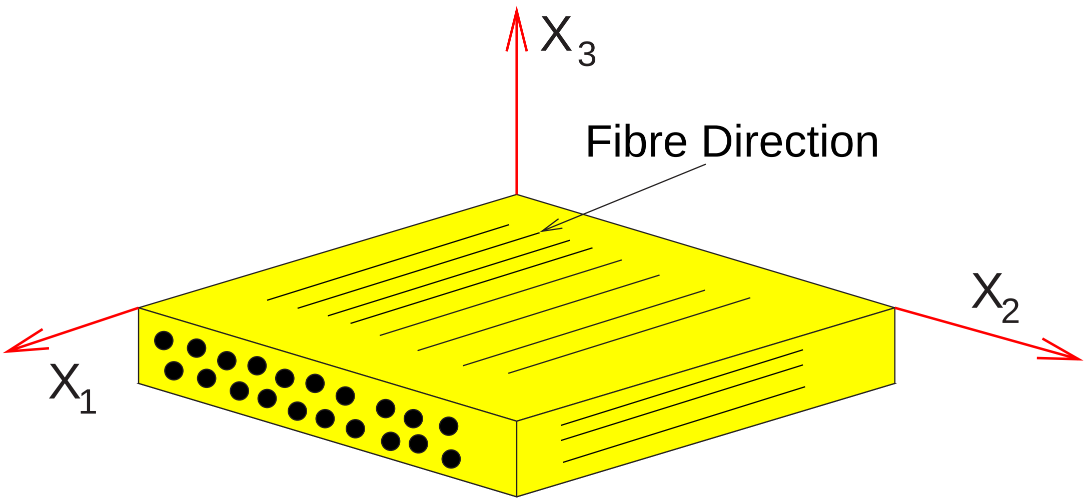
$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{bmatrix}$$




# 18 Plates – Simplifications

## 18.1 Transversely Isotropic

If laminate is in fact transversely isotropic (unidirectional laminate), and in plane strain, we get the same expression relating stress and strain as we do for an orthotropic plate. This applies if the fibres are aligned with the plane of the plate as shown in the fibure below.




Note that the decoupling of shear/extensional stress/strain is crucial to this step. The plane of the plate has to be closely related to the principle directions of the laminate.